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GENERATION OF INTERNAL WAVES BY BOTTOM ROUGHNESS OF THE INTERFACE OF TWO FLUIDS FLOWING AT ANGLES TO EACH OTHER

I. V. Sturova

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The simplest example of three-dimensional internal waves in a stream whose velocity varies with depth in both magnitude and direction is waves on the interfacial surface of two fluids of different densities flowing at an angle to one another. Investigation of the kinematic characteristics of the wave motion in such a fluid under the condition that the depth of the lower layer is infinite was performed in [1]. The asymptotic behavior of waves on the interfacial surface that occur during the flow around a body for the case of infinitely deep layers and of an obstacle on the bottom under the condition of infinite thickness of the upper layer was examined in [2]. The stability of waves occurring on the interfacial surface of two infinite streams flowing at an angle to each other was investigated in [3].

Let us consider the flow around an elevation described by the function f(x, z), by a stream infinite in the horizontal directions, in whose upper layer of thickness H_1 the fluid density is ρ_1, while it is $\rho_2 = \rho_1(1 + \varepsilon)$ $(\varepsilon \geqslant 0)$ in the lower layer of thickness H_2. The velocity of the lower stream is U_2 and is along the x axis, while the velocity of the upper stream is U_1 and makes an angle α with the x axis. The x and z axes are on the unperturbed interfacial surface, the y axis is vertically upward, and the axis of symmetry of the obstacle passes through the origin.

Assuming the fluid motion within each layer to be irrotational, and the perturbations on the free surface and the interfacial surface to be small, we write the equations for the velocity potentials of the perturbed motion in each layer in the form

$$\Delta\varphi_1 = 0 \text{ for } 0 \leqslant y \leqslant H_1,\ \Delta\varphi_2 = 0 \text{ for } -H_2 \leqslant y < 0 \quad (1)$$

with boundary conditions on the free surface ($y = H_1$)

$$\partial\varphi_1/\partial y + L_1\zeta = 0,\ L_1\varphi_1 = g\zeta; \quad (2)$$

on the interfacial surface ($y = 0$)

$$\partial\varphi_1/\partial y + L_1\eta = 0,\ \partial\varphi_2/\partial y + L_2\eta = 0,\ \rho_2 L_2\varphi_2 - \rho_1 L_1\varphi_1 = g(\rho_2 - \rho_1)\eta; \quad (3)$$

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on the bottom ($y = -H_2$)

$$\partial\varphi_2/\partial y + L_2 f = 0,\ \varphi_1,\ \varphi_2 \to 0 \ \text{ as } \ x^2 + z^2 \to \infty,$$

where

$$L_1 \equiv U_1(\cos\alpha\cdot\partial/\partial x + \sin\alpha\cdot\partial/\partial z),\ L_2 \equiv U_2\partial/\partial x.$$

Here the functions $\zeta(x, z)$ and $\eta(x, z)$ describe the vertical displacements of the free surface and the interfacial surface, respectively, and g is the acceleration of gravity. Often the simpler condition of a "solid cover," for which $\partial\varphi_1/\partial y = 0$ at $y = H_1$, is used in place of (2) in investigations of internal waves on the free surface.

Let us introduce the dimensionless variables by taking the quantities $h = f(0, 0)$ (height of the elevation) and U_2 as the scalar units of the length and velocity, and by using the Fourier transform

$$\varphi_*(\mu, y, \nu) = \int_{-\infty}^{\infty} e^{-i\mu x}dx \int_{-\infty}^{\infty} e^{-i\nu z}\varphi(x, y, z)\,dz$$

for real μ and ν, we obtain a system of ordinary differential equations from (1), whose solution will yield the following representations for the functions $\zeta_*(\mu, \nu)$ and $\eta_*(\mu, \nu)$, the Fourier transforms of the functions ζ and η:

$$\zeta_* = \frac{4k^2 d_1^2 d_2^2 f_* (1+\varepsilon)\, e^{-k(H_1+H_2)}}{\left(1+e^{-2kH_1}\right)\left(1+e^{-2kH_2}\right)D},$$

$$\eta_* = \frac{2kd_2^2 f_* (1+\varepsilon)\, e^{-kH_2}}{\left(1+e^{-2kH_2}\right)D}\left(kd_1^2 - \Lambda\,\mathrm{th}\,kH_1\right),$$

where

$$D = \Lambda D_1 - kd_1^2 D_2;$$

$$D_1(k, \theta) = \left[\varepsilon\Lambda\,\mathrm{th}\,kH_2 - (1+\varepsilon)kd_2^2\right]\mathrm{th}\,kH_1 - kd_1^2\,\mathrm{th}\,kH_2;$$

$$D_2(k, \theta) = \left(\varepsilon\Lambda - kd_1^2\,\mathrm{th}\,kH_1\right)\mathrm{th}\,kH_2 - (1+\varepsilon)kd_2^2;$$

$$d_1 = V\sin(\theta+\alpha);\quad d_2 = \sin\theta;\quad \Lambda = gh/U_2^2;\quad V = U_1/U_2;$$

f_* is the Fourier transform of the function $f(x, z)$ and the substitution

$$\mu = k\sin\theta,\ \nu = k\cos\theta$$

is performed. The function η_* becomes

$$\eta_* = -\frac{2kd_2^2 f_* (1+\varepsilon)\, e^{-kH_2}\,\mathrm{th}\,kH_1}{\left(1+e^{-2kH_2}\right)D_1}$$

when the condition of a "solid cover" is used on the free surface.

Executing the inverse Fourier transform, we obtain

$$\eta(x, z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} e^{i\mu x}d\mu \int_{-\infty}^{\infty} e^{i\nu z}\eta_* d\nu = \frac{1}{2\pi^2}\,\mathrm{Re}\int_0^{\pi} d\theta \int_0^{\infty} k e^{ikr\sin(\theta+\varphi)}\eta_* dk, \qquad (4)$$

where the substitution $x = r\cos\varphi,\ z = r\sin\varphi$ has been performed. An analogous expression will also hold for the function $\zeta(x, z)$.

The functions ζ_* and η_* have simple poles that are the roots of the equation $D(k, \theta) = 0$ or $D_1(k, 0) = 0$. It can be seen that for

$$\Lambda\left[V^2 H_2 \sin^2(\theta+\alpha) + H_1\sin^2\theta\right] - V^4\sin^2\theta\sin^2(\theta+\alpha) < \frac{\varepsilon\Lambda H_1 H_2}{1+\varepsilon}$$

the equation $D(k, \theta) = 0$ has two positive roots $\bar{k}_{1,2}$ ($\bar{k}_1 < \bar{k}_2$), otherwise just one root $\bar{k}_2$ exists. The equation $D_1(k, \theta) = 0$ has not more than one positive root $\bar{k}_1$ and only under the condition

$$V^2 H_2 \sin^2(\theta+\alpha) + (1+\varepsilon)H_1\sin^2\theta < \varepsilon\Lambda H_1 H_2$$

We use the Rayleigh method of introducing small dissipative forces [1] proportional to the fluid particle velocities to select the contour of integration in the k-plane in (4). Hence, only the dynamical conditions on the free surface in (2) and on the interfacial surface in (3) undergo changes in the initial problem (1)-(3), and they will now have the form

$$L_1\varphi_1 + \beta\varphi_1 = g\zeta,\ \rho_2 L_2\varphi_2 - \rho_1 L_1\varphi_1 + \beta(\rho_2\varphi_2 - \rho_1\varphi_1) = g(\rho_2 - \rho_1)\eta,$$

where $\beta > 0$ is the dissipation factor which is small in magnitude. The solution of (1) with these boundary conditions shows that the poles of the integrand in (4) have the form $k = \bar{k} + i\beta\gamma$ as $\beta \to 0$, where for the condition of a "solid cover," e.g.,

$$\gamma = \frac{2\bar{k}_1\left[V\sin(\theta+\alpha)\operatorname{cth}\bar{k}_1H_1 + (1+\varepsilon)\sin\theta\operatorname{cth}\bar{k}_1H_2\right]}{\varepsilon\Lambda - \bar{k}_1^2\left[(1+\varepsilon)H_2\sin^2\theta/\operatorname{sh}^2\bar{k}_1H_2 + V^2H_1\sin^2(\theta+\alpha)/\operatorname{sh}^2\bar{k}_1H_1\right]}.$$

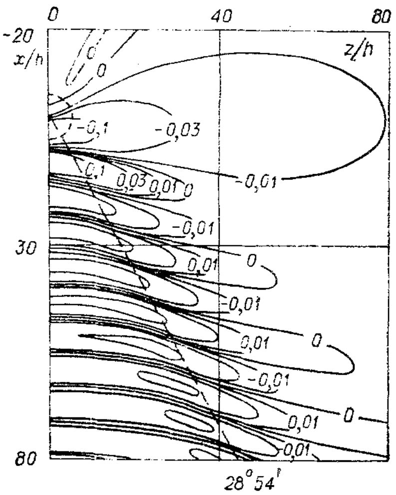

Fig. 1

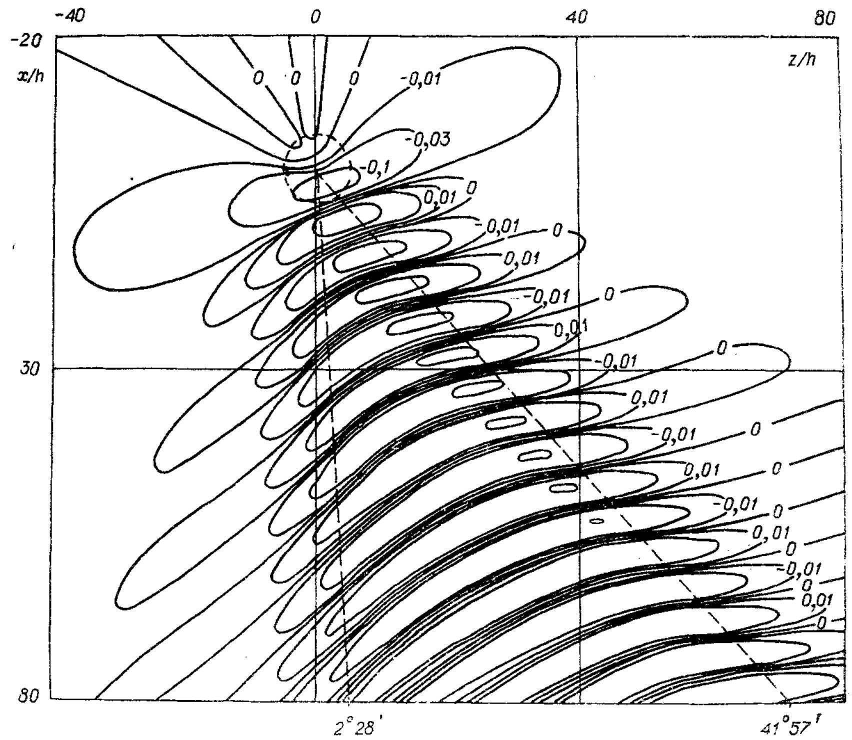

Fig. 2

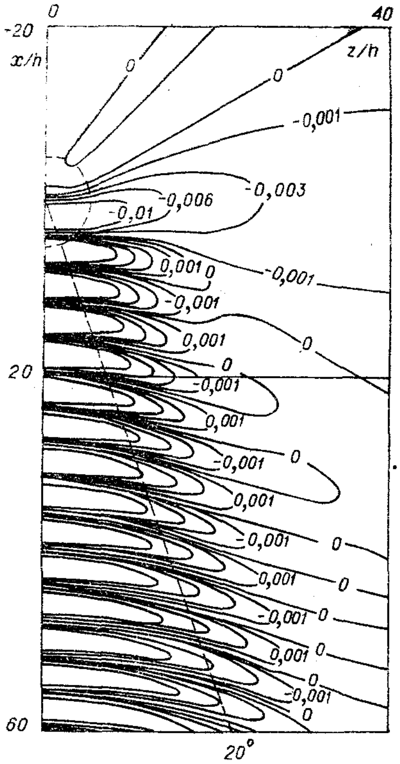

Fig. 3

Therefore, the contour of integration in (4) is selected in the first or fourth quadrant, depending on the sign of $\sin(\theta + \varphi)$, where all the real poles are bypassed by small semicircles on which Im k < 0 for $\gamma > 0$ and Im k > 0 for $\gamma < 0$. This circumstance was not taken into account in [2] and all the poles were bypassed from below.

Consequently, the integral representations for the desired functions can be written as the sums of single integrals because of the presence of poles and of double integrals occuring because of integration along the imaginary axis, which will later be excluded from consideration since they describe local effects in the neighborhood of the elevation and decrease rapidly with the growth of r.

The final expression for the function $\eta(r, \varphi)$ has the form (analogously for the function $\zeta(r, \varphi)$)

$$\eta(r, \varphi) = -\frac{1}{\pi}\sum_{j=1}^{2}\int_{a_j}^{b_j} \bar{k}_j \sin(\bar{k}_j \sin(\theta + \varphi)) \operatorname{Res} \eta_* (\bar{k}_j, \theta)\, d\theta,$$

where $\bar{k}_j$ is the root of the equation $D(k, \theta) = 0$ or $D_1(k, \theta) = 0$, and the integration is over those ranges of values θ in which $\operatorname{sgn}(\sin(\theta + \varphi)) = \operatorname{sgn}(\gamma)$.

In executing the specific computations the shape of the axisymmetric elevation on the bottom was given in two forms

$$f(r) = 1 - r^2/d^2 \text{ for } 0 \leqslant r \leqslant d,\ f(r) = 0 \text{ for } d < r < \infty, \quad (5)$$

$$f_*(k) = 4\pi J_2(kd)/k^2,$$

where J_2 is the Bessel function of second order of the first kind,

$$f(r) = d^3/(d^2 + 2r^2)^{3/2} \text{ for } 0 \leqslant r < \infty, \quad (6)$$

$$f_*(k) = \pi d^2 \exp(-dk/2)/2,$$

for which the volumes of both elevations agree and equal $S = \pi d^2/2$. The case of the flow around a dipole located on the bottom at the point x = 0, z = 0 with axis along the x axis

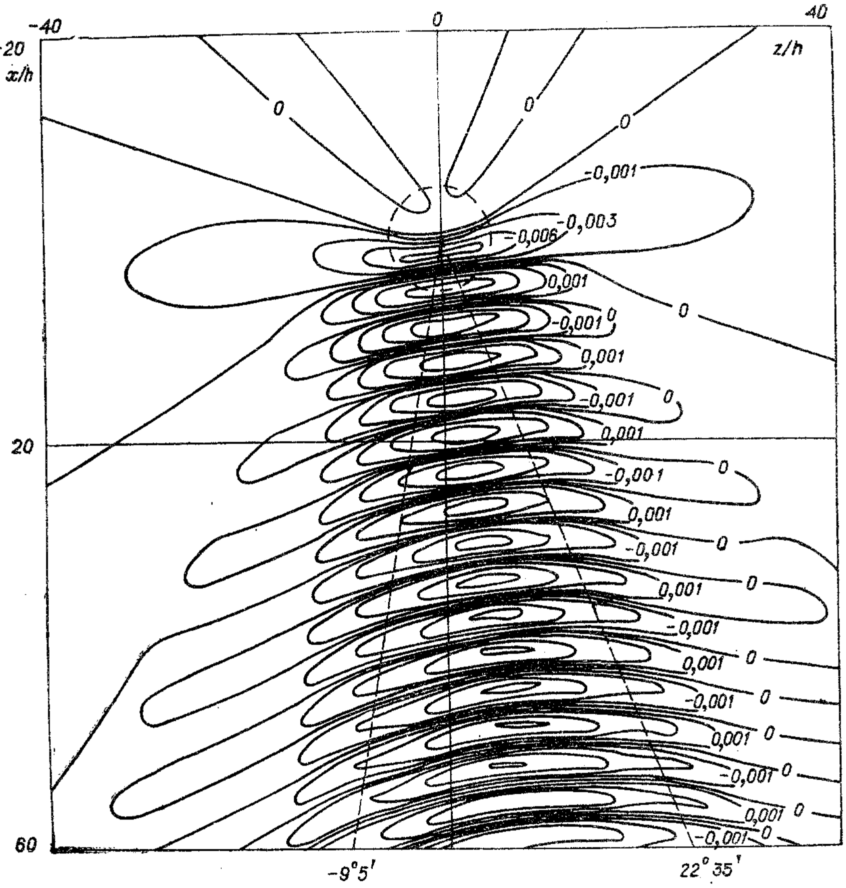

Fig. 4

and momentum equal to S is also investigated for comparison. In this case $f_* = S$. As is known (cf. [4]), such an approximation is used in investigating internal waves generated by a moving body.

Numerical computations of the function $\eta(x, z)$ were performed for two values of V and α for $\Lambda = 100$, $\varepsilon = 10^{-2}$, $H_1/h = 2$, $H_2/h = 5$. Isolines of the functions $\eta(x, z)/h$ are represented in Figs. 1-4 for $V = 1$, $\alpha = 0.45°$ (Figs. 1 and 2) and $V = 0.5$, $\alpha = 0.45°$ (Figs. 3 and 4) for an obstacle described by the function (5) for d = 5. The boundaries of the wave zones determined by the method of stationary phase analogously to [2] are shown by dashes. The circle with center at the origin corresponds to the boundary of the elevation. For $\alpha = 0$ the flow map is symmetric relative to the x axis. The isolines in Figs. 1 and 2 are from the following levels: 0, $\pm 10^{-2}$, $\pm 3 \cdot 10^{-2}$, $\pm 10^{-1}$, $\pm 3 \cdot 10^{-1}$, and in Figs. 3 and 4 from 0, $\pm 10^{-3}$, $\pm 3 \cdot 10^{-3}$, $\pm 6 \cdot 10^{-3}$, $\pm 10^{-2}$, $\pm 3 \cdot 10^{-2}$.

It should be noted that for given values of the initial parameters, the difference between the solutions obtained by using complete conditions on the free boundary and the "solid cover" condition does not exceed 1%. The vertical displacements on the free surface are several orders of magnitude less than on the interfacial surface.

An investigation of the influence of the obstacle shape on the internal wave amplitude showed that the dipole approximation yields exaggerated values. The function $\eta(x, z)/h$ is represented in Fig. 5 as a function of x for $z/h = 20$ and the above-mentioned values of Λ, ε, H_1, H_2 for $V = 1$ (a — $\alpha = 0$, b — $\alpha = 45°$). Curves 1-3 correspond to elevations given by (5), (6) and the dipole approximation. For $V = 0.5$ and different α, the wave amplitudes for both kinds of elevations become quite similar; however, their divergence from the dipole approximation is still more substantial than for $V = 1$.

The stability of waves occurring on the interfacial surface of two infinite streams directed at an angle to each other was investigated in [3]. The results of this paper are easily extended to the case of streams of finite depth upon compliance with the "solid cover" condition on the free surface. Waves with the wave number k will be stable if

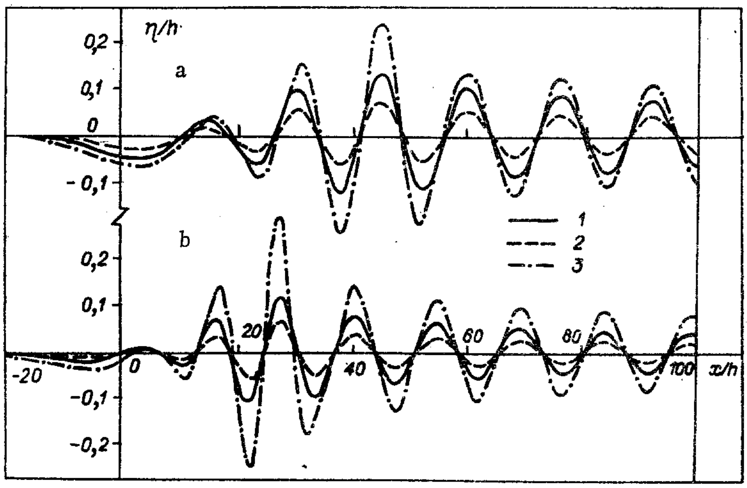

Fig. 5

$$\frac{k}{(1+\varepsilon)\,\mathrm{th}\,kH_1 + \mathrm{th}\,kH_2} < \frac{\Lambda\varepsilon}{(1+\varepsilon)\,[V\sin(\theta+\alpha) - \sin\theta]^2}.$$

The wave motion obtained for the stationary problem examined above will always be stable.

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APPARENT INTERNAL WAVES IN A FLUID WITH EXPONENTIAL DENSITY DISTRIBUTION

S. A. Makarov and Yu. D. Chashechkin

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On the basis of a modified stationary phase method proposed in [1, 2], the constant phase surfaces of internal waves excited by a body moving at an arbitrary angle to the horizon, which agree satisfactorily with those observed experimentally, are determined in [3] in the plane and three-dimensional cases. Taking account of the integral transforms [4, 5], the plane and spatial problems of wave motions occurring during the flow around submerged sources and sinks of identical intensity by a uniform fluid stream stratified with respect to the density are considered by numerical methods in a linear formulation in [6]. The asymptotic solution for the wave field excited by a dipole and an arbitrary source-sink system moving in an exponentially stratified fluid is obtained in [7, 8]. These solutions describe the wave pattern occurring during motion of a body at high velocities.

The purpose of this paper is the determination of the amplitude phase characteristics of apparent internal waves in a fluid with an exponential density distribution for uniform horizontal body displacement in a broad range of motion regimes (including motion at low velocities) and their subsequent comparison with the results of laboratory experiments. Dissipative and diffusion effects (i.e., the change in particle density during motion is not

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